Big balls and small balls

— A trial to explain solving simultaneous equations to children —

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Abstract

One day I chatted with my friend. She told me her children took a test of mathematics and we told about it. But one thing was missing, which was how to explain it to children. This article is such a trial. Mathematically, the problem is solving simultaneous equations, or solving a linear system. However, you do not need any of those knowledge to read this text.

1 The problem

> We have big balls and small balls. The price of five small balls and three big balls is 5.1 Euro. Five big balls and three small balls is 6.26 Euro. How much are a small ball and a big ball? See Figure 1.

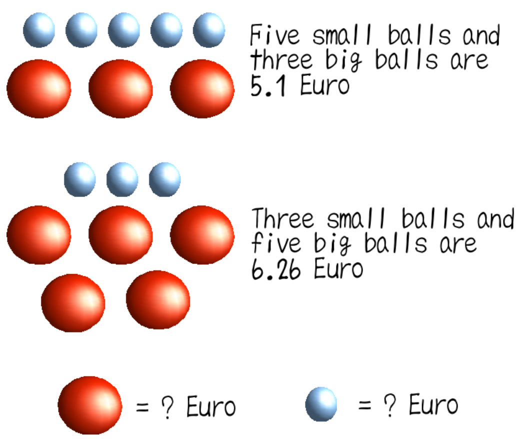

Figure 1: A big balls and small balls problem.

Let's say, a big ball's price is b and a small ball's price is s. The notation does not matter, we could use the big ball's price is a and the small ball's price is b, but I like b and s, since b reminds me a big ball, and s reminds me a small ball.

We have already the first equation, but no need to be scared.

$$5s + 3b = 5.1 \tag{1}$$

We could **read** this as five small balls ($5 \times s$) and three big balls ($3 \times b$) equals 5.1 Euro. Same as the other one.

$$3s + 5b = 6.26 \tag{2}$$

We could also read this as three small balls ($3 \times s$) and five big balls ($5 \times b$) equals 6.26 Euro.

Notation may be a bit strange, but first of all, Mathematicians are lazy. Computer scientists are also lazy, they work hard to not to work. The work should be done by a computer automatically. Here, $\times$ is not written and 'the price of a small ball' is written as s, all of them are for being lazy as possible, but, importantly, without loss of clearness. All the information are still kept. That's the reason we did not remove the '+' here. We can only remove '$\times$.'

2 Why the problem is difficult?

If this problem is so easy, then we do not need to continue this article. But, I do not see this is so easy. The problem is difficult because there are two unknowns, one is big ball's price, and the other is small ball's price.

If we only have small balls, the problem is simpler. For example, we have 10 small balls and total cost is 50 Euro. Then one small ball's price is $50/10 = 5$ Euro.

The problem comes from two unknowns. Here, one of the most important idea is how to solve the difficult problem. There is a great idea to solve a difficult problem. The Roman empire used this idea often. It called “Divide and Conquer.”

When the Roman empire faced a large enemy, they first divide it to half and conquer one by one. If the half is still too large as an enemy, then they divide the enemy again, until one becomes enough small to conquer.

This idea is quite general and we can use it for many things. For example, when I eat a big pizza, I can not put it whole to my mouth, but then cut it enough small size and eat one by one. Then I can eat a big pizza.

We could use this idea to solve this problem. We have two unknowns, it is to difficult to solve. But if we can remove one of the unknown, we can conquer that.

But how can we remove one enemy (= unknown)? There is a great mathematical tool called **Euclid's axioms**. It is general, powerful, and amazingly so simple to understand. Sometimes, even people think this is too obvious. Yes, indeed. But, sometimes people do not know such a simple thing can do so great.

3 Euclid's axioms

There are many Euclid's axioms. Here we use one of them. Remember, the purpose of using this axiom is to remove one of the unknown of our equations.

- Euclid's axiom of “Element”: If equals are added to equals, then the sums are equal. See Figure 2.

This means, if I have ten Euro and you have ten Euro (equals), and if we both get 5 Euro (another equals to be added), we both have the same amount (We both have $10+5 = 15$ Euro.) Simple, isn't it? If we will go to really strict, there are still some problems to be considered, but, I think for this problem, this is fine.

The multiplication is a repeat of the sum. For example, $2 + 2 + 2 = 3 \times 2$. Therefore, we could use this axiom with the multiplication shown in Figure 3.

If two children both can carry two litters milk (equals) at once from a store to their house, and both three times did it, means multiply 3 (multiply both equal times), they both have the same amount ($2 \times 3 = 6$ litters milk) at their home.

So far, I explained about this tool, next we use this tool to divide the problem, remove one unknown.

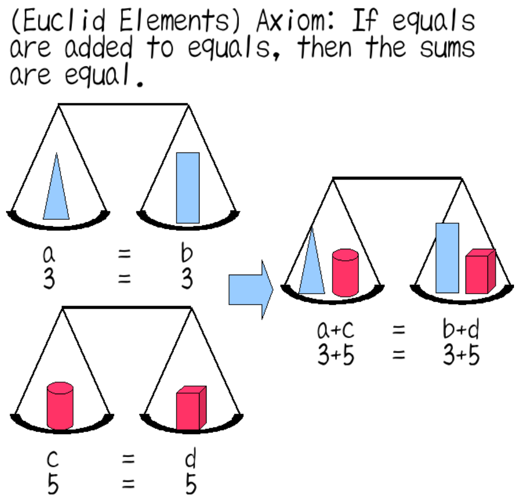

Figure 2: One of Euclid's axioms of Element.

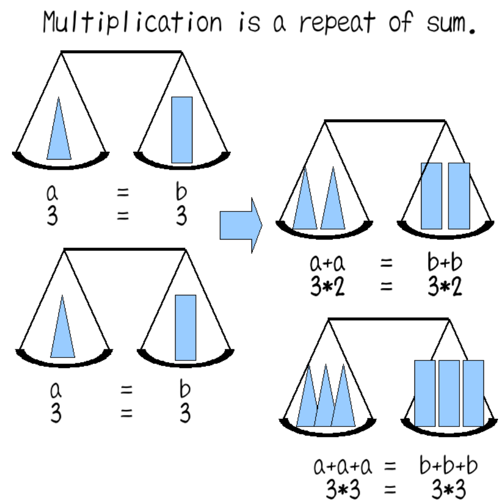

Figure 3: Apply it to multiplication since multiplication is a repeat of sum.

4 Use the tool.

Before attack the problem, think about what is an equation. An equation is a relationship between two things, which is equal. Therefore it is called equations. Our Equation (1) shows left hand side ($5s + 3b$) and right hand side (5.1) are equal. The keyword here is equal. Euclid's axiom is also about the relation between two equals. We have: (1) equals, (2) a tool for manipulating equals. That's why we told about the axiom.

Let's practice to use the tool a bit. If we double the both sides of Equation (1), we will get the next equation.

$$\begin{aligned} 2 \times (5s + 3b) &= 2 \times 5.1 \\ 10s + 6b &= 10.2 \end{aligned}$$

Same as the three times,

$$\begin{aligned} 3 \times (5s + 3b) &= 3 \times 5.1 \\ 15s + 9b &= 15.3 \qquad (3) \end{aligned}$$

Is it OK? It is just both sides are equal, then we multiply the same equals to both sides. This point is sometimes hard. However, we use only one tool so far.

We could apply the axiom to Equation (2) also, since it is equation. Here, we will do it 5 times in some reason, which will be clear soon.

$$\begin{aligned} 5 \times (3s + 5b) &= 5 \times 6.26 \\ 15s + 25b &= 31.3 \qquad (4) \end{aligned}$$

Let's compare the Equation (3) and Equation (4).

$$\begin{aligned} \underline{15s} + 9b &= 15.3 \\ \underline{15s} + 25b &= 31.3 \end{aligned}$$

The underlined terms are the same. This is what we want. Because we can use the axiom also in subtraction – If equals are subtracted by equals, then the rests are equal.[1]

Now, this is tricky. Equation (3)'s left hand side and right hand side are equal as well as Equation (4)'s left hand side and right. We could use the axiom again. If the both sides are equal – Equation (4)'s both sides are equal, we can subtract an equal from the both side. And Equation (3)'s both sides are equal.

Can we do that? Here, always the tool is the same, Euclid's axiom. Now we are applying a bit more. An equation minus an equation.

Equation (4) - Equation (3):

$$\begin{aligned} \text{(4)'s left} - \text{(3)'s left} &= \text{(4)'s right} - \text{(3)'s right} \\ 15s + 25b - (15s + 9b) &= 31.3 - 15.3 \\ \underbrace{15s - 15s}_{vanish!} + 25b - 9b &= 16 \end{aligned}$$

Yes! We have removed one unknown. The rest is easy.

$$\begin{aligned} 16b &= 16 \\ b &= 1 \end{aligned}$$

[1] At the Euclid time (around B.C 300), minus number is not clearly known. So, Euclid did not tell about subtraction. But we know now a subtraction is 'adding a minus number.'

We got the big ball's price 1 Euro.

Now we know the one answer. Use this answer in Equation (1),

$$\begin{aligned} 5s + \underline{3b} &= 5.1 \quad \text{(big ball=1 Euro,} 3b = 3) \\ 5s + 3 &= 5.1 \\ 5s + 3 - 3 &= 5.1 - 3 \quad \text{(axiom again)} \\ 5s &= 2.1 \\ s &= 0.42 \end{aligned}$$

The small ball's price is 42 cent.

Using the axiom again and again, remove an unknown term. One question is how to choose the multiplication number. We choose Equation (1) for three, Equation (2) for five. Actually, this is least common multiple (das kleinste gemeinsame Vielfache), but this would be a little bit too much for a day. So, we will close this story here.

5 Summary

We solved a problem using two ideas.

- Divide and conquer: Divide a difficult problem to small and simple several problems until we can solve each of them. Here we remove one unknown first, not directory solve two unknowns simultaneously.
- Euclid's axiom from Element: If equals are added to equals, then the sums are equal.

6 And beyond

- Generalize this problem: What happens if there are middle size balls. Can we solve such problem also?
- Linear system and Matrix: Yes, under some condition, we can solve more unknowns. When we can solve, how can we solve, are all in this topic.
- Linear programming: What happens if they are not equations. For example, there are two kind of investment A and B. Under several constraint, tax, interest, Freistellungsauftraege, ... how to maximize the interest and minimize the tax?

I hope you enjoy this article.

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